Adapted Hedging*

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Abstract

Exponentials of squared returns in Gaussian densities, with their consequently thin tails, are replaced by the absolute return to form Laplacian and exponentially tilted Laplacian densities at unit time. Scaling provides densities at other maturities. Stochastic processes with these marginals are identified. In addition to a specific local volatility model the densities are consistent with the difference of compound exponential processes taken at log time and scaled by the square root of time. The underlying process has a single parameter, the constant variance rate of the process. Delta hedging using Laplacian and Asymmetric Laplacian implied volatilities are developed and compared with Black Merton Scholes implied volatility hedging.The hedging strategies are implemented for stylized businesses represented by dynamic volatility indexes. The Laplacian hedge is seen to be smoother for the skew trade. It also performs better through the financial crisis for the sale of strangles. The Laplacian and Gaussian models are then synthesized as special cases of a model allowing for other powers between unity and the square. Numerous hedging strategies may be run using different powers and biases in the probability of an up move. Adapted strategies that select the best performer on past quarterly data can dominate fixed strategies. Adapted hedging strategies can effectively reduce drawdowns in the marked to market value of businesses trading options.

1 Introduction

The Black-Scholes (1973) and Merton (1973) model for pricing options is widely employed in quoting option prices indirectly through the use of the Black-Merton-Scholes implied volatility. These implied volatilities are then employed

*Much of the work on the theoretical structure of the model in this paper was done in collaboration with Marc Yor in the summer of 2001. We are greatly indebted to his contributions. Marc Yor passed away January 9, 2014

to infer the option delta or derivative of option price with respect to a movement in the spot price. Aggregating across all deltas for open option positions an aggregate delta is determined and a position opposite to the delta is taken in the spot asset as a hedge. This practice is called delta hedging at the Black-Merton-Scholes implied volatility.

The risk neutral density of the Black-Merton-Scholes model for the logarithm of the stock is a Gaussian density that is symmetric and very thin tailed with probabilities falling exponentially in the square of return. Actual risk neutral densities in markets reflect significant levels of skewness and much fatter tails reflected in the implied volatility smile. It is therefore a reasonable conjecture that hedging performance may possibly be improved by employing a skewed and heavier tailed distribution while keeping the structure of a one parameter model permitting fast access to such an implied parameter. With a view towards organizing finite moments of all orders we investigate here, first the use of a Laplacian distribution that falls linearly with the exponential of the absolute return as opposed to its square. As a second step we allow for asymmetry by differentiating the rate of probability decay in the tails for positive and negative returns. Later we synthesize both the Laplacian and Gaussian models as special limiting cases allowing for other powers between unity and the square. This latter model is referred to here as the Laplace Gauss (LG) model.

For the Laplacian model the result is an option pricing model with exponential tail probabilities that are even simpler than the cumnorm function embedded in the Black-Merton-Scholes model. The LG model requires numerical approximation methods. The single implied parameter in all cases is again the volatility of the distribution with fast access to both the implied volatility and the associated delta. An adapted hedging strategy can be built where one employs the best hedge based on a performance evaluation of a variety of hedges on the immediate past data. For an effective comparison of hedging performance we implement delta hedging for businesses that take various positions in options daily, that then daily delta hedge all open positions and daily mark all open positions to market. We thereby construct an index representing the value of the business accumulating all related cash flows along with marking all open positions to market. The indexes are constructed for real market data from May 1997 to April of 2015 in some cases and from March 15 2006 to March 16 2016 in other cases. The daily change in the post hedge index value represents the random shocks to net worth for the business attained with different hedge designs.

A conservative valuation of the business that aggregates risk and reward is then determined as the bid price for the associated business risk by evaluating an expectation with respect to a non-additive probability exaggerating the probability of losses while simultaneously depressing the probability of gains. Such valuations are characteristic of valuations in two price economies (Madan (2012), (2015a)) defining acceptable risks as a convex cone of random variables containing the nonnegative random variables (Artzner, Delbaen, Eber and Heath (1999)). Hedges delivering higher bid prices are examples of superior hedging mechanisms. The bid prices may also be seen as reward less risk with reward being the usual additive expectation and risk being the ask price for negative of the centered random variable. This is because once one has centered the random variable it is pure risk that requires the purchase of its negative to exit the risk. The cost of such a purchase is the ask price for the negated and centered random variable. Such a formulation for the bid price is developed in Madan (2015b). One may rank hedging strategies based on the bid price for daily changes in recent net worth and then implement the hedge for the next day that has the highest bid price. We refer to such a hedge as an adapted hedging strategy.

We observe that for businesses trading and holding short maturity deep out of the money options the different hedging strategies are comparable. However when near money options are involved then the Laplacian hedge may provide a superior hedge. We perform conservative valuations of businesses by evaluating the expectation with respect to a nonadditive probability of cash flows and or changes in net worth generated by the business over the next year. We observe that hedging assists value when open positions are marked to market and hence one may argue that the primary motivation for hedging may be the culture of marking open positions to market. Hedging may reduce the value of cash flows in the absence of marking open positions to market. Laplacian or LG hedging is also observed to deliver better hedges in times of financial crisis or when markets are more volatile. Adapted hedging can provide substantial improvements over any fixed or predetermined hedging strategy.

For the Laplacian model, the stochastic process being employed for the logarithm of the stock price is shown to be consistent with a stochastic differential equation that incorporates a local volatility model in which volatilities rise as the spot deviates from its expected value in either direction at a uniform and constant response rate. There is also a one dimensional Markovian martingale with independent and inhomogeneous increments that has the adopted Laplacian densities at each maturity. In fact the discontinuous process consistent with these marginals is the homogeneous difference of compound exponential Poisson processes evaluated at $\ln(t)$ and scaled by $\sqrt{t}$.

After a presentation of the details on the Laplacian model, the two models Laplacian and Gaussian, are then made special cases of a more general model allowing for arbitrary powers other than unity and the square to lead to the Laplace Gauss, LG model. For both the Laplacian and LG models we entertain asymmetric versions of the model permitting some skew adjustments in the delta hedge design.

Section 2 presents the densities, the formulas for European call and put option prices along with the deltas needed for delta hedging for the Laplacian model. The underlying stochastic processes consistent with the Laplacian model are presented in section 3. Section 4 takes up the case of an Asymmetric Laplacian model. Section 5 presents a sample of implied volatility curves for the three models. Section 6 presents a comparison for hedging businesses using delta hedging at Black-Merton-Scholes implied volatilities with those obtained from the Laplacian and Asymmetric Laplacian models. Section 7 presents comparisons for various option selling businesses and their conservative valuation time paths. In Section 8 we synthesize the Laplacian and Black Merton Scholes

Gaussian models to formulate the LG model. Experiments with LG implied volatilities and deltas are reported on in Section 9. Section 10 develops and implements Adapted Hedging. Section 11 concludes.

2 The Return Densities

The Gaussian return density is symmetric about zero with tails that decline with the exponential of the square of the return. Pricing densities are known to have fatter tails, or tails that decline at rates below that of the Gaussian. We consider instead densities with tails that decline with the exponential of the absolute value of the return, as opposed to its square. This structure is comparable to the form of the variance gamma Lévy densities, where we also have predominantly exponential decay rates. Consider therefore the one parameter family

$$g(x,t) = \frac{1}{\sigma\sqrt{2t}} \exp\left(-\frac{\sqrt{2}\,|x|}{\sigma\sqrt{t}} \right), \quad -\infty < x < \infty \tag{1}$$

This density, like the Gaussian, is symmetric with zero odd moments and one may easily compute its variance to be $\sigma^2 t$. We also verify by the change of variable $y = x/\sqrt{t}$ that

$$\begin{aligned} h(y) &= \frac{1}{\sigma\sqrt{2}} \exp\left(-\frac{\sqrt{2}\,|y|}{\sigma} \right) \\ &= g(y,1) \end{aligned}$$

and hence the proposed densities satisfy the same scaling condition as the original Black-Merton-Scholes model.

The model for the stock price under Black-Merton-Scholes for a constant continuously compounded interest rate of r and a dividend yield of q, is given by

$$S(t) = S(0) \exp\left((r-q)t + Z(t) - \frac{\sigma^2 t}{2} \right)$$

where $Z(t)$ has the Gaussian density of zero mean and variance $\sigma^2 t$.

We replace the variable $Z(t)$ by $X(t)$ that has the density of equation (1) and accordingly alter the exponential compensation on evaluating the expectation

$$
\begin{aligned}
E\left[\exp\left(X_t\right)\right] &= \int_{-\infty}^{\infty} e^x \frac{1}{\sigma\sqrt{2t}} \exp\left(-\frac{\sqrt{2}\,|x|}{\sigma\sqrt{t}}\right) dx \\
&= \int_0^{\infty} \frac{1}{\sigma\sqrt{2t}} \exp\left(-\left(\frac{\sqrt{2}}{\sigma\sqrt{t}}+1\right)x\right) dx + \int_0^{\infty} \frac{1}{\sigma\sqrt{2t}} \exp\left(-\left(\frac{\sqrt{2}}{\sigma\sqrt{t}}-1\right)x\right) \\
&= \frac{1}{\sigma\sqrt{2t}}\left(\frac{1}{\frac{\sqrt{2}}{\sigma\sqrt{t}}+1}+\frac{1}{\frac{\sqrt{2}}{\sigma\sqrt{t}}-1}\right) \\
&= \frac{1}{\sigma\sqrt{2t}}\left(\frac{\frac{2\sqrt{2}}{\sigma\sqrt{t}}}{\frac{2}{\sigma^2 t}-1}\right) \\
&= \frac{1}{\left(1-\frac{\sigma^2 t}{2}\right)} = \exp\left(-\log\left(1-\frac{\sigma^2 t}{2}\right)\right)
\end{aligned}
$$

Hence we write the stock price at time t as

$$
S(t) = S(0)\exp\left((r-q)t + X(t) + \log\left(1-\frac{\sigma^2 t}{2}\right)\right). \tag{2}
$$

2.1 European Call and Put Option Pricing

The prices of European Call and Put options consistent with the specification (2) are easily computed to be

$$
\begin{aligned}
C(K) &= S(0)\exp(-qt)\frac{\exp(-\left(\sqrt{2}-\sigma\sqrt{t}\right)|d|)}{2}\left(1+\sigma\sqrt{t/2}\right) - K\exp\left(-rt\right)\frac{\exp\left(-\sqrt{2}\,|d|\right)}{2},\ d>0 \\
P(K) &= K\exp\left(-rt\right)\frac{\exp\left(-\sqrt{2}\,|d|\right)}{2} - S(0)\exp(-qt)\frac{\exp(-\left(\left(\sqrt{2}+\sigma\sqrt{t}\right)|d|\right)}{2}\left(1-\sigma\sqrt{t/2}\right),\ d<0 \\
C(K) &= P(K)+S(0)\exp(-qt)-K\exp(-rt),\ d<0 \\
P(K) &= C(K)+K\exp(-rt)-S(0)\exp(-qt),\ d>0. \\
d &= \frac{\log(K/S(0))}{\sigma\sqrt{t}} - \left(\frac{r-q}{\sigma}\right)\sqrt{t} - \frac{\log(1-\frac{\sigma^2 t}{2})}{\sigma\sqrt{t}}.
\end{aligned}
$$

These price relations may be inverted to derive the appropriate implied volatilities.

2.2 Construction of Deltas

The call and put deltas may be computed as follows:

For $d>0$ the call price is

$$
\frac{1}{2}Se^{-qt}e^{-(\sqrt{2}-\sigma\sqrt{t})d}(1+\sigma\sqrt{t/2}) - \frac{1}{2}Ke^{-rt}e^{-\sqrt{2}d}
$$

It follows that

$$\begin{aligned}\frac{\partial C}{\partial S} &= \frac{1}{2}e^{-qt}e^{-(\sqrt{2}-\sigma\sqrt{t})d}(1+\sigma\sqrt{t/2}) \\ &+\frac{1}{2}Se^{-qt}e^{-(\sqrt{2}-\sigma\sqrt{t})d}(1+\sigma\sqrt{t/2})(\sigma\sqrt{t}-\sqrt{2})\frac{\partial d}{\partial S} \\ &+\frac{1}{2}Ke^{-rt}e^{-\sqrt{2}d}\sqrt{2}\frac{\partial d}{\partial S}\end{aligned}$$

We now substitute from the definition of d that

$$Ke^{-rt} = Se^{-qt}e^{\sigma\sqrt{t}d}(1-\frac{\sigma^2 t}{2})$$

We then obtain that the sum of the last two terms is

$$\frac{1}{2}Se^{-qt}e^{-(\sqrt{2}-\sigma\sqrt{t})d}\left[(\frac{\sigma^2 t}{2}-1)\sqrt{2}+\sqrt{2}(1-\frac{\sigma^2 t}{2})\right]\frac{\partial d}{\partial S} = 0$$

It follows that

$$\frac{\partial C}{\partial S} = \frac{1}{2}e^{-qt}e^{-(\sqrt{2}-\sigma\sqrt{t})d}(1+\sigma\sqrt{t/2}) \tag{3}$$

Similarly for $d < 0$ we have that the put price is given by

$$\frac{1}{2}Ke^{-rt}e^{\sqrt{2}d} - Se^{-qt}e^{(\sqrt{2}+\sigma\sqrt{t})d}\left(1-\sigma\sqrt{t/2}\right)$$

It follows that

$$\begin{aligned}\frac{\partial P}{\partial S} &= -\frac{1}{2}e^{-qt}e^{\sqrt{2}(\sqrt{2}+\sigma\sqrt{t})d}\left(1-\sigma\sqrt{t/2}\right) \\ &+\frac{1}{2}Ke^{-rt}e^{\sqrt{2}d}\sqrt{2}\frac{\partial d}{\partial S} \\ &-\frac{1}{\sqrt{2}}Se^{(\sqrt{2}+\sigma\sqrt{t})d}\left(1-\sigma\sqrt{t/2}\right)(\sqrt{2}+\sigma\sqrt{t})\frac{\partial d}{\partial S}\end{aligned}$$

Once again substituting for Ke^{-rt} we obtain that the sum of the last two terms is

$$\frac{1}{2}Se^{-qt}e^{\sqrt{2}(1+\sigma\sqrt{t})d}\left[\sqrt{2}(1-\frac{\sigma^2 t}{2})-\sqrt{2}(1-\frac{\sigma^2 t}{2})\right]\frac{\partial d}{\partial S} = 0$$

Hence we have that

$$\frac{\partial P}{\partial S} = -\frac{1}{2}e^{-qt}e^{(\sqrt{2}+\sigma\sqrt{t})d}\left(1-\sigma\sqrt{t/2}\right) \tag{4}$$

for $d < 0$.

From put call parity we know that

$$\frac{\partial C}{\partial S} - \frac{\partial P}{\partial S} = e^{-qt}$$

so for $d < 0$ we have that

$$\frac{\partial C}{\partial S} = e^{-qt} + \frac{\partial P}{\partial S} \tag{5}$$

while for $d > 0$

$$\frac{\partial P}{\partial S} = \frac{\partial C}{\partial S} - e^{-qt} \tag{6}$$

Equations (3, 5, 4, 6) complete the delta calculations for the Laplacian pricing model.

3 Underlying Stochastic Processes for Laplacian densities

We develop two representations for the continuous time stochastic process consistent with the marginal densities associated with the stock price model (2). The first is a continuous process for the logarithm of the stock price. The second is a representation as a purely discontinuous martingale for the stock price itself.

3.1 A Continuous Representation

We note that the logarithm of the stock price is given by

$$\begin{aligned} \log(S(t)) &= \log(S(0)) + (r-q)t + \log\left(1 - \sigma^2 t\right) + X(t) \\ &= \log(S(0)) + \int_0^t (r-q) - \frac{\sigma^2}{1-\sigma^2 s} ds + X(t) \end{aligned} \tag{7}$$

The process $X(t)$ has the marginal densities given by equation (1) . These are zero mean densities and we seek to write $X(t)$ as a continuous martingale. For this purpose consider the representation

$$X(t) = \int_0^t \sigma(s, X_s) dW(s) \tag{8}$$

We know by construction that the laws of $X(t)$ satisfy the scaling property in that for any fixed $c > 0$

$$(X_{ct}, t \geq 0) \overset{law}{=} \left(\sqrt{c} X_t, t \geq 0\right) \tag{9}$$

and furthermore we know that

$$g(x,t) = \frac{1}{\sqrt{t}} h\left(\frac{x}{\sqrt{t}}\right).$$

It is shown in Madan and Yor (2002) that the continuous martingale representation (8) for such marginals is obtained on defining

$$\begin{aligned} \sigma^2(s,x) &= \overline{a}\left(\frac{x}{\sqrt{s}}\right) \\ \overline{a}(y) &= \frac{1}{h(y)}\int_y^\infty zh(z)dz \end{aligned}$$

In our particular case we may determine the function $\overline{a}(y)$ on performing the integration with respect to the function h. The result is given by

$$\overline{a}(y) = \sigma^2 + \sigma\,|y|\,.$$

It follows that we have the continuous representation

$$X(t) = \int_0^t \sqrt{\sigma^2 + \sigma\left|\frac{X_s}{\sqrt{s}}\right|}dW(s)$$

Substituting for $X(t)$ in terms of $S(t)$ from equation (7) we may write the stochastic differential equation for the log of the stock price directly as

$$\begin{aligned} \log(S(t) &= \log(S(0)) + \int_0^t \left[(r-q) - \frac{\sigma^2}{1-\sigma^2 u}\right] du + \\ & \int_0^t \sigma\sqrt{1 + \frac{1}{\sigma\sqrt{u}}\left|\log(S(u)/S(0)) - (r-q)u - \log\left(1-\sigma^2 u\right)\right|}dW(s) \end{aligned}$$

We observe that the local volatility is linear in the absolute value of the deviation of log prices from their mean measured in standardized units. Hence this model builds in some symmetric local volatility that is probably correct for the put side, given the relative flatness of implied volatilities in this direction. For the call side the climb is probably too steep, requiring one to lower the implied volatilities.

3.2 A Discontinuous Representation

We may also represent our densities as resulting from a discontinuous Markov inhomogeneous martingale.

This is seen as follows. Consider the process

$$L(a) = \ell_{T(a)}$$

where $\ell(s)$ is the local time at zero of an independent Brownian motion and $T(a)$ is the first passage time of this Brownian motion to the level a. Hence $L(a)$ is the local time at zero of a Brownian motion upto the first passage time of this Brownian motion to the level a.

It is well known that $L(a)$ is an exponential random variable with a mean of $2a$ (See for example Revuz and Yor Chapter XIII). It follows that

$$E\left[\exp\left(-\lambda L(a)\right)\right] = \frac{1}{1+2a\lambda}$$

We may now compute the characteristic function for an independent Brownian motion evaluated at $L(a)$, $Y(a) = B(L(a))$ and observe that

$$E\left[\exp\left(iuY(a)\right)\right] = \frac{1}{1+au^2}$$

Computing the characteristic function of the density $g(x,t)$ we see that that

$$E\left[\exp\left(iuX(t)\right)\right] = \frac{1}{1+\sigma^2 tu^2}$$

It follows that

$$X(t) \overset{law}{=} B\left(L\left(\sigma^2 t\right)\right)$$

and we have the representation of $X(t)$ as an inhomogeneous Markov Lévy process. Since the law of $L(\sigma^2 t)$ is exponential we also have the structure of a taking the variance of a normal random variable to be exponentially distributed. For such a process we have that

$$\begin{aligned} M(t) &= \frac{\exp\left(X(t)\right)}{E\left[\exp\left(X(t)\right)\right]} \\ &= \exp\left(X(t) + \log\left(1-\sigma^2 t\right)\right) \end{aligned}$$

is a martingale and

$$S(t) = S(0)\exp\left((r-q)t\right)M(t).$$

It follows that the Futures price

$$\begin{aligned} F(t,T) &= S(t)e^{(r-q)(T-t)} \\ &= S(0)e^{(r-q)T}M(t) \end{aligned}$$

is a martingale. In fact it is a purely discontinuous martingale. For a unit starting value we may write tha

$$F(t,T) = (e^x - 1) * (\mu - \nu)$$

where μ is the integer valued random measure associated with the jumps of $X(t)$ and ν is its predictable compensator.

To determine ν, we note that

$$\nu(dx,dt) = k(x,t)dxdt$$

where

$$\frac{1}{1+\sigma^2 u^2 t} = \exp\left(\int_0^t \int_{-\infty}^{\infty} \left(e^{iux} - 1\right) k(x,s)dxds\right).$$

Equivalently we may write

$$-\log\left(1+\sigma^2u^2t\right) = \int_0^t\int_{-\infty}^{\infty}\left(e^{iux}-1\right)k(x,s)dxds$$

Differentiating with respect to t we get that

$$-\frac{\sigma^2u^2}{1+\sigma^2u^2t} = \int_{-\infty}^{\infty}\left(e^{iux}-1\right)k(x,t)dx$$

Now differentiate with respect to u to get that

$$-\frac{2\sigma^2u}{\left(1+\sigma^2u^2t\right)^2} = i\int_{-\infty}^{\infty}e^{iux}xk(x,t)dx$$

Hence we have that

$$i\int_{-\infty}^{\infty}\left(\cos(ux)+i\sin(ux)\right)xk(x,t)dx = -\frac{2\sigma^2u}{\left(1+\sigma^2u^2t\right)^2}$$

and as $xk(x,t)$ is antisymmetric

$$\int_{-\infty}^{\infty}\sin(ux)xk(x,t)dx = \frac{2\sigma^2u}{\left(1+\sigma^2u^2t\right)^2}$$

Integrating both sides with respect to u we get that

$$-\int_{-\infty}^{\infty}\cos(ux)k(x,t)dx = -\frac{1}{\left(1+\sigma^2u^2t\right)t}$$

We may then write

$$\begin{aligned}\int_0^{\infty}\cos(ux)k(x,t)dx &= \frac{1}{2\left(1+\sigma^2u^2t\right)t}\\ &= \frac{1}{2\sigma^2t^2\left(\frac{1}{\sigma^2t}+u^2\right)}\\ &= \frac{1}{2\sigma t^{3/2}}\frac{\left(\frac{1}{\sigma\sqrt{t}}\right)}{\left(\frac{1}{\sigma^2t}+u^2\right)}\end{aligned}$$

We now note that

$$\int_0^{\infty}e^{-ax}\cos(ux)dx = \frac{a}{\left(a^2+u^2\right)}$$

It follows that

$$k(x,t) = \frac{\exp\left(-\frac{|x|}{\sigma\sqrt{t}}\right)}{2\sigma t^{3/2}}.$$

Making the change of variable $y = |x|/\sqrt{t}$ to obtaing the Lévy system

$$l(y,t) = \frac{\exp\left(-\frac{y}{\sigma}\right)}{\sigma t}$$

whereby we observe that $X(t)$ is the difference of two independent compound exponential processes evaluated at log time and scaled by $\sqrt{t}$. Let $Y(t)$ be the difference of two independent compound Poisson exponential variates. Then

$$X(t) = \sqrt{t} Y(\ln t).$$

4 The Asymmetric Case

Consider now Brownian motion with drift evaluated at $L(a)$. This gives us

$$X(a) = \theta L(a) + \sigma B(L(a))$$

and the characteristic function of $X(a)$ is

$$\begin{aligned}
& E\left[\exp\left(iuX(a)\right)\right] \\
= & E\left[\exp\left(iu\theta L(a) - \frac{\sigma^2 u^2}{2} L(a)\right)\right] \\
= & E\left[\exp\left(-\left(\frac{\sigma^2 u^2}{2} - iu\theta\right) L(a)\right)\right] \\
= & \frac{1}{1 + 2a\left(\frac{\sigma^2 u^2}{2} - iu\theta\right)} \\
= & \frac{1}{1 - 2iu\theta a + \sigma^2 u^2 a}
\end{aligned}$$

Let us now compare this with the density that has the form

$$g(x,t) = \begin{cases} ce^{-b|x|} & x < 0 \\ ce^{-ax} & x > 0 \end{cases}$$

To organize an integral of unity we must have

$$c\left(\frac{1}{a} + \frac{1}{b}\right) = 1$$

so

$$c = \left(\frac{1}{a} + \frac{1}{b}\right)^{-1}.$$

The characteristic function is

$$\begin{aligned}
& \int_0^\infty e^{-iux} ce^{-bx} dx + \int_0^\infty e^{iux} ce^{-ax} dx \\
= & c\left(\frac{1}{b + iu} + \frac{1}{a - iu}\right) \\
= & \frac{c(a+b)}{ab + (a-b)iu + u^2}
\end{aligned}$$

4.1 Calibrating the asymmetry

Under the proposed form the area $x > 0$ is

$$\frac{\frac{1}{a}}{\frac{1}{a}+\frac{1}{b}} = \frac{1}{1+\frac{a}{b}}$$

Let the probability that $x > 0$ be η. It follows that

$$b = a\frac{\eta}{1-\eta} \tag{10}$$

From a knowledge of η we may freeze the ratio of b to a by the equation (10). Our density at a particular maturity is then given by

$$g(x) = \begin{cases} a\eta e^{-ax} & x > 0 \\ a\eta e^{-a\eta|x|/(1-\eta)} & x < 0 \end{cases}$$

We next determine the variance of this density. For this we note that

$$\begin{aligned} \int_0^\infty x^2 e^{-cx} dx &= \int_0^\infty \frac{w^2}{c^3} e^{-w} dw \\ &= \frac{2}{c^3} \end{aligned}$$

It follows that

$$\begin{aligned} \int_{-\infty}^\infty x^2 g(x) dx &= \frac{2(1-\eta)^3}{a^2\eta^2} + \frac{2\eta}{a^2} \\ &= \frac{2}{a^2}\left[\eta + \frac{(1-\eta)^3}{\eta^2}\right] \end{aligned}$$

Also we have that as

$$\int_0^\infty x e^{-cx} dx = \frac{1}{c^2}$$

that

$$\int_{-\infty}^\infty x g(x) dx = \frac{1}{a}\left[\frac{\eta^2 - (1-\eta)^2}{\eta}\right]$$

Hence the variance is

$$\frac{1}{a^2}\left(\frac{(1-\eta)^2 + \eta^2}{\eta^2}\right)$$

If we set this to $\sigma^2 t$ we obtain that

$$a = \frac{\sqrt{(1-\eta)^2 + \eta^2}}{\sigma\eta\sqrt{t}}$$

The numerator is unity at $\eta = 0, 1$. For $\eta = 1/2$, we have that $a = \frac{\sqrt{2}}{\sigma\sqrt{t}}$, the result with the symmetry case. This minimum value for the numerator is $1/\sqrt{2}$ and it occurs at $\eta = 1/2$.

The density is then given by

$$
\begin{aligned}
g(x,t) &= \begin{cases} \frac{\alpha}{\sigma\sqrt{t}} e^{-\frac{\alpha x}{\sigma\eta\sqrt{t}}} & x > 0 \\ \frac{\alpha}{\sigma\sqrt{t}} e^{-\frac{\alpha|x|}{\sigma(1-\eta)\sqrt{t}}} & x < 0 \end{cases} \\
\alpha &= \sqrt{\eta^2 + (1-\eta)^2} \\
\eta &= P(x > 0).
\end{aligned}
$$

4.2 The asymmetric Stock Price Model

For the convexity correction we need to evaluate

$$
\begin{aligned}
& \int_{-\infty}^{\infty} e^x g(x,t) dx \\
= & \int_0^{\infty} \frac{\alpha}{\sigma\sqrt{t}} e^x e^{-\frac{\alpha x}{\sigma\eta\sqrt{t}}} dx + \int_0^{\infty} \frac{\alpha}{\sigma\sqrt{t}} e^{-x} e^{-\frac{\alpha x}{\sigma(1-\eta)\sqrt{t}}} dx \\
= & \frac{\alpha}{\sigma\sqrt{t}} \left[\frac{1}{\frac{\alpha}{\sigma\eta\sqrt{t}} - 1} + \frac{1}{\frac{\alpha}{\sigma(1-\eta)\sqrt{t}} + 1} \right] \\
= & \frac{\alpha}{\sigma\sqrt{t}} \left[\frac{\sigma\eta\sqrt{t}}{\alpha - \sigma\eta\sqrt{t}} + \frac{\sigma(1-\eta)\sqrt{t}}{\alpha + \sigma(1-\eta)\sqrt{t}} \right] \\
= & \frac{\alpha}{\sigma\sqrt{t}} \left[\frac{\alpha\sigma\sqrt{t}}{(\alpha - \sigma\eta\sqrt{t})(\alpha + \sigma(1-\eta)\sqrt{t})} \right] \\
= & \exp\left[-\log(1 - \frac{\sigma\eta\sqrt{t}}{\alpha}) - \log(1 + \frac{\sigma(1-\eta)\sqrt{t}}{\alpha}) \right]
\end{aligned}
$$

This stock price model is therefore

$$
\begin{aligned}
S(t) &= S(0) e^{(r-q)t + \sigma\sqrt{t}x + \omega} \\
\omega &= \log\left(1 - \frac{\sigma\eta\sqrt{t}}{\alpha}\right) + \log\left(1 + \frac{\sigma(1-\eta)\sqrt{t}}{\alpha}\right)
\end{aligned}
$$

where the random variable x has the density

$$
h(x) = \begin{cases} \alpha e^{-\frac{\alpha}{\eta}x} & x > 0 \\ \alpha e^{-\frac{\alpha}{1-\eta}|x|} & x < 0 \end{cases}
$$

4.3 Call and Put Option Prices

We determine the call and put prices as before based on values of $d > 0$ or $d < 0$. The equation for d is

$$\begin{aligned} d &= \frac{\log\left(\frac{K}{S(0)}\right)}{\sigma\sqrt{t}} - \left(\frac{r-q}{\sigma}\right)\sqrt{t} - \frac{\log\left(1-\frac{\sigma\eta\sqrt{t}}{\alpha}\right)}{\sigma\sqrt{t}} - \frac{\log\left(1+\frac{\sigma(1-\eta)\sqrt{t}}{\alpha}\right)}{\sigma\sqrt{t}} \\ \alpha &= \sqrt{\eta^2+(1-\eta)^2} \\ \eta &= \Pr(x>0). \end{aligned}$$

For $d > 0$ the call price is

$$\begin{aligned} C(K) &= S(0)e^{-qt}F_{1c}(d) - Ke^{-rt}F_{2c}(d) \\ F_{1c}(d) &= \eta\left(1+\frac{\sigma(1-\eta)\sqrt{t}}{\alpha}\right)e^{-\frac{\alpha}{\eta}(1-\frac{\sigma\eta\sqrt{t}}{\alpha})|d|} \\ F_{2c}(d) &= \eta e^{-\frac{\alpha}{\eta}|d|} \end{aligned}$$

For $d < 0$ the put price is

$$\begin{aligned} P(K) &= Ke^{-rt}F_{2p}(d) - S(0)e^{-qt}F_{1p}(d) \\ F_{1p}(d) &= (1-\eta)\left(1-\frac{\sigma\eta\sqrt{t}}{\alpha}\right)e^{-\frac{\alpha}{1-\eta}\left(1+\frac{\sigma(1-\eta)\sqrt{t}}{\alpha}\right)|d|} \\ F_{2p}(d) &= (1-\eta)e^{-\frac{\alpha}{1-\eta}|d|} \end{aligned}$$

For $d < 0$ we get call by put call parity and similarly for $d > 0$ we get put by put call parity.

4.4 Further remarks on calibrating asymmetries

We observe that the logarithm of the forward price at maturity relative to the initial forward price is positive if the forward price exceeds $F(0)\exp(\omega)$ where ω is the convexity correction for the model and $F(0)$ is the initial forward price. Hence by pricing a digital call option on the forward at the strike given by $\exp(\omega)$ one may estimate η the probability $x > 0$. This may then be used to calibrate the asymmetry. However, such a procedure may get noisy when evaluating deep out of the money digital calls and this could hinder the performance of the hedge. It may be better to allow for a variety of the smaller asymmetries and adapt the hedge to evaluating its performance on data from the recent past. A later section presents results on such adapted hedging.

4.5 Asymmetric Case Deltas

The deltas for the asymmetric case are given as follows.

For $d > 0$ we get the call delta as

$$\begin{aligned}\frac{\partial C}{\partial S} &= e^{-qt}F_{1c}(d) + \\ & Se^{-qt}F_{1c}(d)\left(\frac{-\alpha}{\eta}\right)\left(1 - \frac{\sigma\eta\sqrt{t}}{\alpha}\right)\frac{\partial d}{\partial S} \\ & -Ke^{-rt}F_{2c}(d)\left(\frac{-\alpha}{\eta}\right)\frac{\partial d}{\partial S}\end{aligned}$$

Now employ as usual the equation

$$Ke^{-rt} = Se^{-qt}e^{\sigma\sqrt{t}d}(1 - \frac{\sigma\eta\sqrt{t}}{\alpha})(1 + \frac{\sigma(1-\eta)\sqrt{t}}{\alpha})$$

and observe that the last terms cancel. We then have the usual construction of delta

$$\begin{aligned}\frac{\partial C}{\partial S} &= e^{-qt}F_{1c}(d);\ d > 0 \\ \frac{\partial P}{\partial S} &= -e^{-qt}F_{1p}(d);\ d < 0 \\ \frac{\partial C}{\partial S} &= e^{-qt} + \frac{\partial P}{\partial S};\ d < 0 \\ \frac{\partial P}{\partial S} &= \frac{\partial C}{\partial S} - e^{-qt};\ d > 0.\end{aligned}$$

5 Sample of Implied Volatilities

For option prices on the $S\&P$ 500 index as at June 8, 2015 we present for four maturities three implied volatility curves. These are for the Black Merton Scholes, Laplacian and Asymmetric Laplacian models. Figure 1 presents these implied volatilities. The Laplacian and Asymmetric Laplacian implied volatilities bend at the forward reflecting the shift in the density.

6 Hedging at Laplacian Implied Volatilities

Comparing hedging strategies in a relevant manner is a difficult task for one has to decide a number of factors in an interesting and meaningful way. In fact the absence of good answers to such questions led us to abandon the presentation of this model back in 2001 when it was first developed it. The factors at issue are, i) what are the instruments being hedged, ii) what is the environment in which the hedge is being conducted and iii) how are the post hedge risks to be evaluated and compared. We may begin by briefly considering some traditional answers to such questions and their associated reservations.

We could consider as instruments a call or put option. For the environment we could take an underlying geometric Brownian motion or other Lévy process

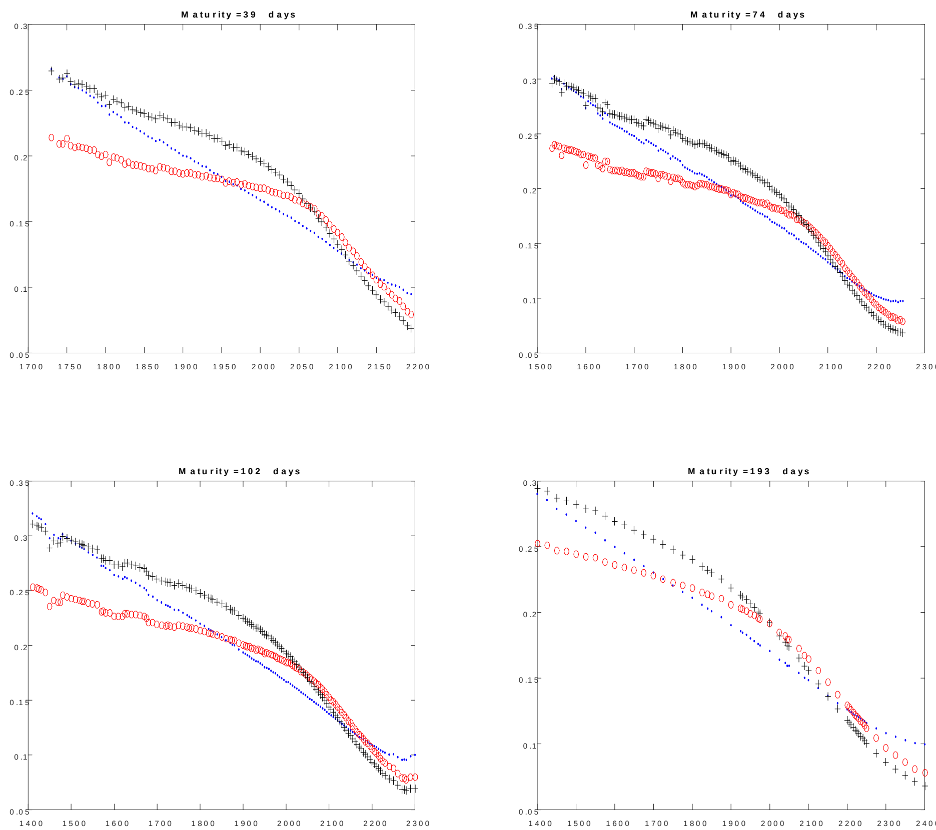

Figure 1: Shown are the Black Merton Scholes implied volatilities in dots. The Variance Exponential implied volatilities are represented by circles and the Asymmetric Variance Exponential implied volatilities are shown by plus signs.

admitting some jumps or discontinuities. For a richer environment one may further add features of stochastic and/or local volatility. The post hedge residual risks may be evaluated using variance as a measure of residual risk. The reservations with these answers are as follows.

A single option is a short lived experience coming into existence briefly for a few months and hence the interest in such an activity is equally limited. When the environment is some model, however sophisticated, the interest is limited by the model as we are certain that we shall not in reality experience any of these models. In this regard we note the analysis of Crépey (2004) that argues for smile adjusted deltas in markets with physical and risk neutral negative skewness. Earlier related comparisons for delta hedging include Bakshi, Cao and Chen (1997), Dumas, Fleming and Whaley (1998) and Vähämaa (2004). With regard to smile adjustments one may observe that given the right risk neutral density, and one, with a positive volatility, exists by no static arbitrage (Breeden and Litzenberger (1978)), then for this density the smile would be flat with no adjustment needed. Hence smile adjustments are just a consequence of using the implied volatility from the wrong density. By exploring other densities we naturally mitigate smile issues. Additionally, what one really wants to know is how the hedges work in real markets over extended periods of time on real positions.

A popular criterion in hedging studies is variance reduction with analytical work by Schweizer (2001) and empirical investigations by Alexander and Barossa (2007). More recently Hull and White (2016) propose extensions of vega adjusted deltas for variance minimization extending the approaches of Alexander, Rubinov, Kalepky and Leontsinis (2012). But we may note with regard to variance reduction that the focus on risk reduction however measured is not a rational economic objective. The risk can be eliminated by not taking any exposure in the first place. The attention should be on the market value of the activity being conducted and the activity of buying or selling a single option is not interesting in the first place. What one may really may want to look at is the market value of on going businesses that embed different hedging systems. Real businesses are of course far too complicated to redo in different ways to then compare results. An interesting solution is offered by the activity of constructing dynamic volatility indices. We consider three here that have Bloomberg tickers of MSCBEVOG, MSUSMSDS and MSUSSPVP. We consider stylized forms of these indices that we now describe.

The dynamic index MSCBEVOG has a number of world wide indices for underliers that are converted to US dollars using current exchange rates. We restrict attention to just the S&P 500 index. Our stylized index every day sells 25 delta one month puts and calls, marks to market all open positions and delta hedges these positions daily taking a position in the daily change in the index itself. For the index MSUSMSDS we sell daily 10 delta puts and buy 40 delta calls, mark open positions to market and delta hedge by postioning in the daily change in the $S\&P$ 500 index. In both cases all the option trading, marking, and delta hedging is done daily from May 9, 1997 to April 25, 2015. Two delta hedges are implemented using the Black Merton Scholes model and the Laplacian model.

The actual indices trade exchange quoted strikes and maturities by approximating the one month maturity by averaging across neighbouring maturities and approximating strikes by exchange traded strikes closest to the desired or target strikes. For the implementation presented here we summarize the option surface across strike and maturity by the four parameters of the $VGSSD$ model introduced in Carr, Geman, Madan and Yor (2007). This model has been calibrated to option prices with maturities between a month and a year and absolute moneyness below 33% for each day over the stated data period and the parameter file has been stored. We may then use the model to trade and mark all options at model prices for all arbitrary strikes and maturities. Hence we trade and hold the exact one month maturity and the strikes with the exact target deltas. This allows us to simulate the path of stylized dynamic volatility indices through time. The exercise may be extended to other underliers by first building the $VGSSD$ parameter file for the other underliers.

The first two questions are then answered by setting up dynamic business enterprises trading options continuously through time for the instruments to be hedged. These are the open positions in each business each day. The environment is the time path of actual option prices across strikes and maturities as they have been realized in markets through time. The result to be evaluated is the dynamic stream of the change in the net worth of the business as embodied in the time path of the dynamic indices. For this we turn our attention to conservative valuation as it occurs for the two price economies of conic finance.

For the conservative valuation of a random variable in conic finance one replaces expectation by expectation with respect to a non-additive probability. The problem with the use of expectation is that it treats all probabilities equally and does not account for one's lack of experience with probabilities associated with extreme tails events. Conservatism in conic finance objectives is attained by inflating the probabilities of tails loss events while simultaneously depressing the probability of tail gain events. An expectation may be seen to be an integral of all tails stretching across levels of losses and gains. An expectation with respect to non-additive probability is an expectation under a distortion of the original distribution function by a concave distribution function. This conservative valuation can be seen to be an infimum of all expectations taken with respect to all alternative probabilities that are bounded above by the distortion of the original probability. The conservative valuation of a liability is the supremum of all such alternative valuations. The former delivers the bid or lower price of a two price economy while the latter delivers the ask or upper price of a two price economy. One can show that the bid or lower price equals the reward or ordinary expectation less a risk charge that is the ask price for the negated centered random variable. Modulo the expectation one holds the centered random variable and exit requires the purchase of its negative at a cost equal to the asking price.

Such conservative valuations of random variables X were introduced in Cherny and Madan (2009, 2010) using the distortion minmaxvar that employs the distortion

$$\Psi(u) = 1 - \left(1 - u^{\frac{1}{1+\gamma}}\right)^{1+\gamma}$$

For a random sample of outcomes for daily changes in the index value x_n yielding the ordered sequence $x_{(n)}$ the bid or lower price is given by

$$b(X) = \sum_{n=1}^{N} x_{(n)} \left(\Psi\left(\frac{n}{N}\right) - \Psi\left(\frac{n-1}{N}\right) \right).$$

We report on such bid prices for various entitites involved in the evaluation of hedged dynamic indices and comment on the related implications. The stress level γ was 0.075. The results for the two dynamic indices are presented in separate subsections.

6.1 Stylized MSCBEVOG results

Figure 2 presents a graph of the index value when hedged at Black Merton Scholes implied volatilities.

Figure 3 presents the excess of the index when hedged by Laplacian and Asymmetric Laplacian implied volatilities.

The bid price for the unhedged cash flow -1.1227. If we add the hedge but do not mark to market then the bid price drops to -7.5123. The bid price for the cash flow marked to market is -11.9078 but unhedged. The cash flow marked to market and hedged using Black Merton Scholes implied volatilities has a bid price of -19.85. The corresponding figures for Laplacian hedging and asymmetric Laplacian variance exponential hedging are respectively -19.8947 and -19.9171.

6.2 Stylized MSUSMSDS results

Figure 4 presents the graph for the index value of this stylized product as hedged by the Black Merton Scholes implied volatility and the Laplacian models.

We observe that the index paths under Variance Exponential and Asymmetric Variance Exponential hedging are smoother. This reflected in the bid prices for the three hedging procedures that are respectively -5.6971, 0.2060 and 0.1530. Unlike the case for the 25 delta trade that is short maturity out of the money with comparable valuations across hedging strategies the valuations here are quite different.

7 Analysis of businesses selling three month strangles on the S&P 500 index.

This section presents an analysis of the stylized index MSUSSPVP. The index sells three month $20, 30, 40$ and 50 delta strangles that are then delta hedged and marked to market. The data period for the analysis of such businesses

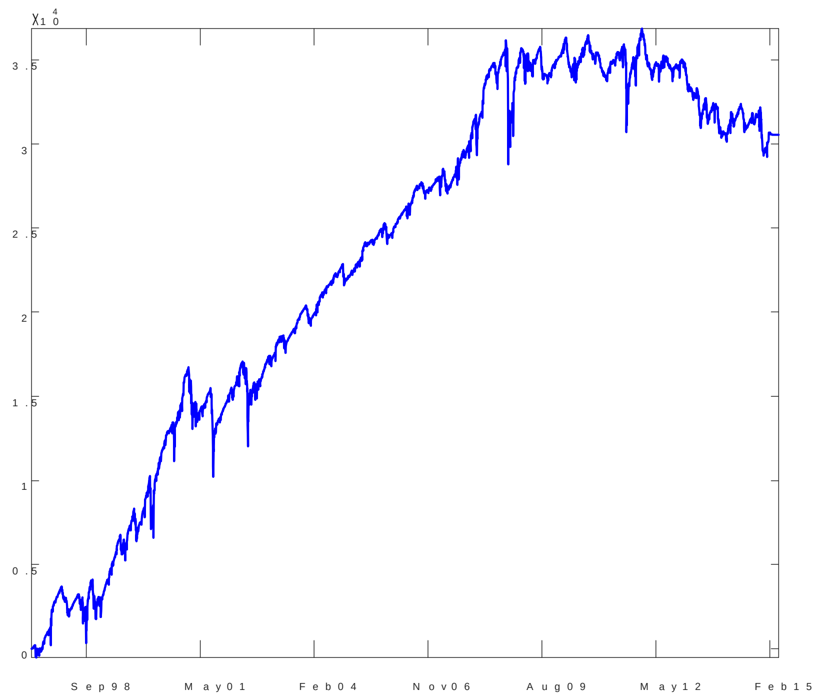

Figure 2: Index selling 25 delta puts and calls.

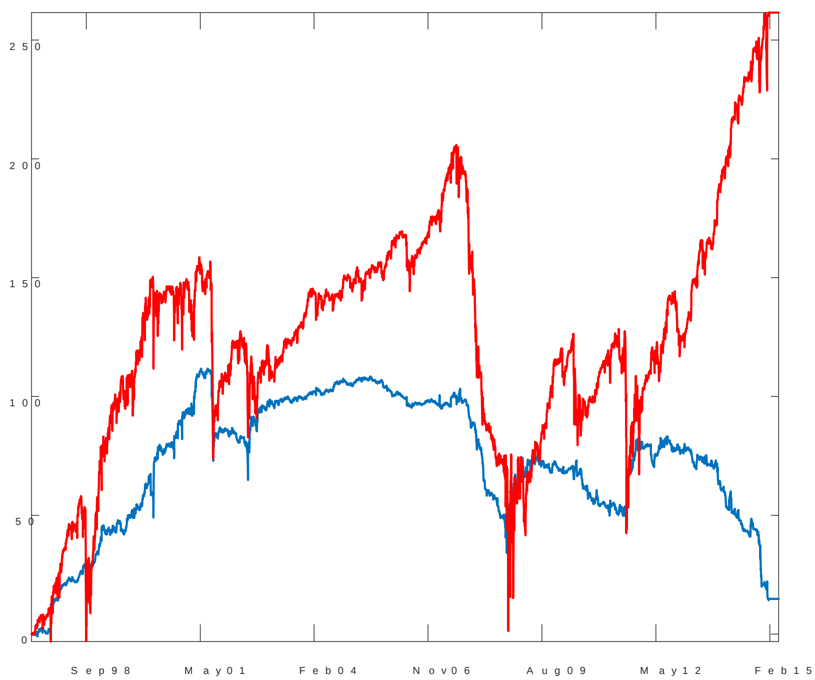

Figure 3: Excess index value from variance exponential and asymmetric variance exponential hedging

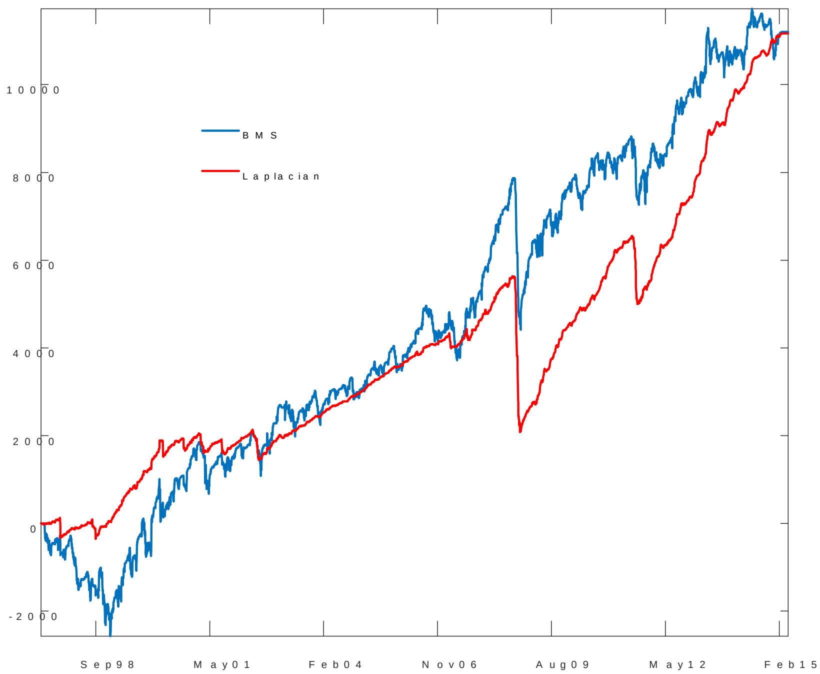

Figure 4: Stylized skew trade hedged by Black Merton Scholes and Laplacian implied volatilities and deltas.

is the level of the S&P 500 index daily from May 9, 1997 to March 3, 2016 along with the parameter file for the $VGSSD$ model calibrated to the option data for maturities between a month and a year. The stylized strategy sells all four strangles daily for an exact three month maturity at the exact deltas and delta hedges and marks to market all open option positions. Cash flows, hedge cash flows and mark to market values are constructed for five businesses, the aggregate business selling all four strangles and the four component businesses that just sell only the 20, 30, 40 and 50 delta strangles. We report here on just the use of the Black Merton Scholes hedge and the Laplacian hedge.

We report results for all five businesses labeled A, B, C, D and E that sell respectively the four strangles, just the 20, 30, 40 and 50 delta strangles. At the stress level of 0.075 we present the conservative valuation or bid prices for the whole period and the last thousand days for the unhedged and unmarkedcash flows CFV, the sum of unhedged cash flow and the daily change in the mark to market value of open positions $CFMV$, the hedged but unmarked cash flows $CFHBSV$ and $CFHVEV$ using Black Merton Scholes hedging or Variance Exponential Hedging and finally the hedged and marked cash flow $CFMHBSV$ and $CFMHVEV$. There are six bid prices reported for each of five businesses for two subperiods each in Table 1. The rows for the diversification benefit DB express the excess of the bid price for the aggregate business A over the sum of the bid prices for the component businesses.

TABLE 1
Bid Prices

Period	Business	CFV	CFMV	CFHBSV	CFHVEV	CFMHBSV	CFMHVEV
	A	55.03	−141.48	−151.10	−181.08	−28.63	−45.14
	B	10.77	−25.12	−28.53	−40.64	−7.44	−18.00
Full	C	13.41	−34.09	−36.54	−46.64	−7.12	−15.30
	D	14.93	−40.62	−42.81	−50.54	−7.76	−11.69
	E	15.52	−43.84	−45.04	−50.42	−8.07	−8.70
	DB	0.40	2.19	1.82	7.16	1.76	8.55
	A	90.86	−80.36	−99.13	−145.09	−26.42	−66.80
Last	B	17.53	−12.02	−17.59	−38.44	−7.17	−26.64
1000	C	22.18	−19.10	−22.89	−39.80	−5.65	−21.77
Days	D	25.02	−24.52	−28.85	−39.83	−7.34	−16.21
	E	25.81	−27.67	−32.11	−37.68	−8.89	−11.14
	DB	0.32	2.95	2.31	10.66	2.63	8.96

We observe from Table 1 that the unhedged and unmarked cash flows from the five businesses are quite valuable. Considerable value is lost by the mark to market and this regained by the hedge in the presence of marking, no matter how it is hedged. In the absence of a mark to market the addition of the hedge is quite detrimental, not matter how the hedge is done. The table clearly supports the position that the benefits of hedging only accrue in the presence of a mark to market culture. All the businesses have better valuations in the last thousand days that exclude the financial crisis present in the full period. The diversification benefits are higher in a Laplacian hedging regime.

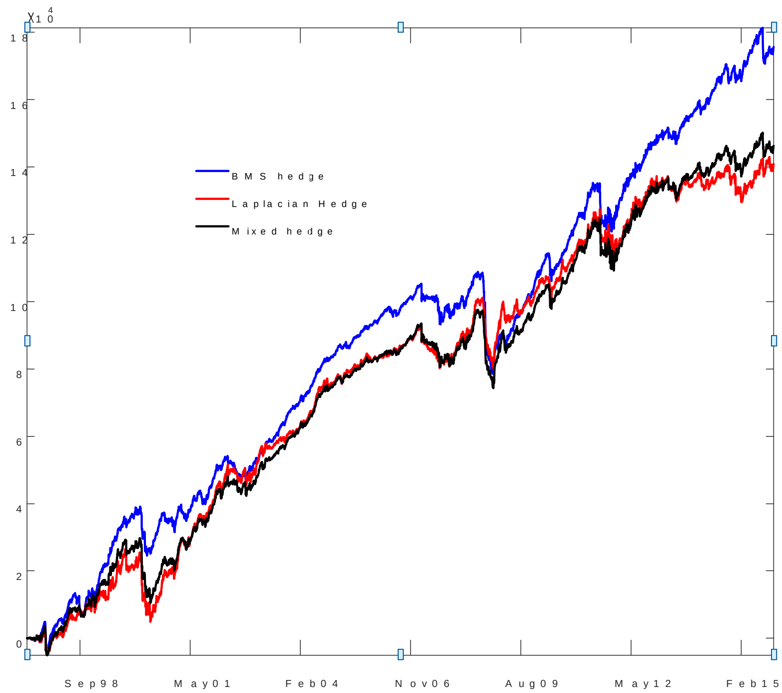

Figure 5: The stylized index MSUSSPVP hedged by Black Merton Scholes, Variance Exponential and a mixture of the two.

Figure 5 presents the time path of the three indices for the aggregate business using Black Merton Scholes hedging, Laplacian hedging and a weighted average of the two deltas.

In light of Table 1 it is instructive at the cummulated cash flows unhedged and unmarked along with just marking, just hedging and both marking and hedging. Figure 6 presents the time paths of these four values.

One may clearly see the volatility induced by the process of marking that is partially compensated by the volatility of hedging resulting in the full index that is both marked and hedged but dominated in all respects by the unmarked and unhedged cash flow. Table 2 presents the performance measures for the four time paths presented in Figure 6 over the last three years with the Max

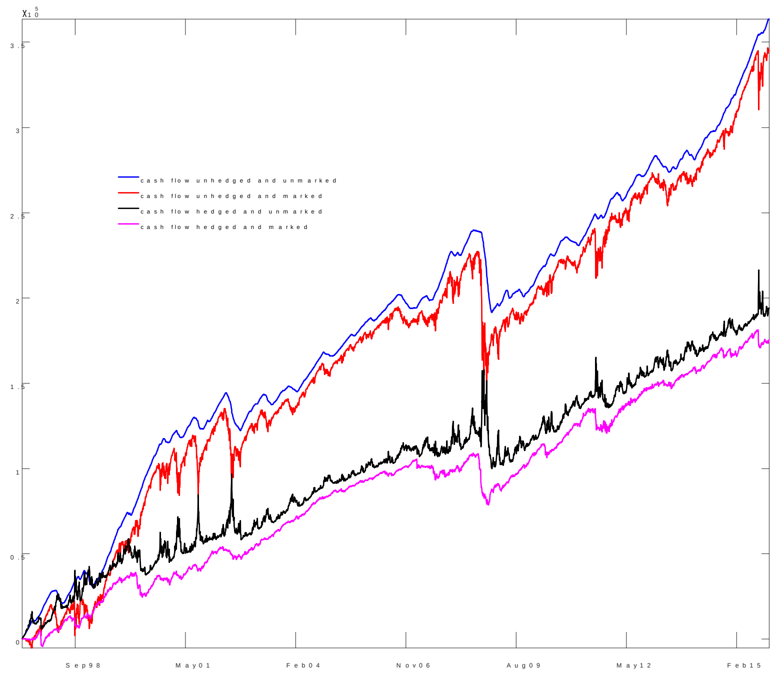

Figure 6: Cummulated cash flows unhedged and unmarked along with just hedging, just marking and both hedging and marking.

Draw Down (MDD) in thousands of dollars.

TABLE 2
MSUSSPVP
Performance Measures

	11/14-11/15				11/13-11/14				11/12-11/13			
	uhum	uhm	hum	hm	uhum	uhm	hum	hm	uhum	uhm	hum	hm
Sharpe	29.67	1.40	0.24	0.65	12.63	1.61	1.10	2.08	4.31	0.11	1.04	1.03
GLR	40.42	1.43	1.07	1.13	5.74	1.33	1.22	1.49	1.84	0.10	1.20	1.21
PP.	0.94	0.70	0.63	0.62	0.82	0.59	0.63	0.68	0.62	0.48	0.62	0.59
AI	1.07	0.06	0.01	0.02	0.48	0.06	0.04	0.08	0.16	0.04	0.04	0.04
MDD	0.476	34.58	27.27	10.58	0.46	10.97	9.08	5.70	9.72	19.36	12.02	4.80
Skw	-1.49	-0.88	-0.10	-2.46	-0.99	-0.18	-0.06	-2.11	-0.37	0.97	-0.11	-0.83
Krt	5.52	24.42	24.39	14.56	3.47	6.08	5.46	11.31	1.89	6.51	5.93	8.05
Peak	0.80	0.89	0.90	0.85	0.70	0.77	0.79	0.81	0.55	0.78	0.76	0.79
Tail	0.05	0.04	0.04	0.04	0.07	0.07	0.07	0.04	0.02	0.06	0.06	0.05

7.1 Valuing the marked and hedged business

A marked to market and hedged business has a cummulated cash position and a marked to market value of open option positions that one may denote by W_t at time t. The daily change $X_t = W_t - W_{t-1}$ may be viewed as the daily PNL_t of the business and we may wish to ask what such a business is worth at time t as an ongoing concern operating in perpetuity. We know the business is risky and the daily PNL consequences are random. If we knew the distribution of this random variable we could place a conservative valuation on the business by using the methods of conic finance by evaluating the bid or lower price of a two price economy. This lower price is an expectation of the random variable taken with respect to a non-additive probability that judiciously raises the probability of tail loss events and lowers the probabilities in the upper tail. With a view to accessing a random sample of daily outcomes we treat the sequence of 252 daily PNL outcomes immediately following time t as such a sample. The valuation at time t is then the distorted expectation of the immediately following 252 daily PNL outcomes. We compute such valuation time paths for the three businesses adopting hedging by Black Merton Scholes and the Laplacian deltas. Figure 7 present a graph of the time path of such valuations stopping 252 days before the end of the data period. We observe a substantial improvement delivered by Laplacian hedging over the period of the financial crisis of 2008.

8 Synthesizing the Laplacian and Gaussian models in the Laplace Gauss (LG) model

We wish to nest the asymmetric Laplace and the Gaussian density as candidates for impled volatility construction followed by delta hedging using derivatives of

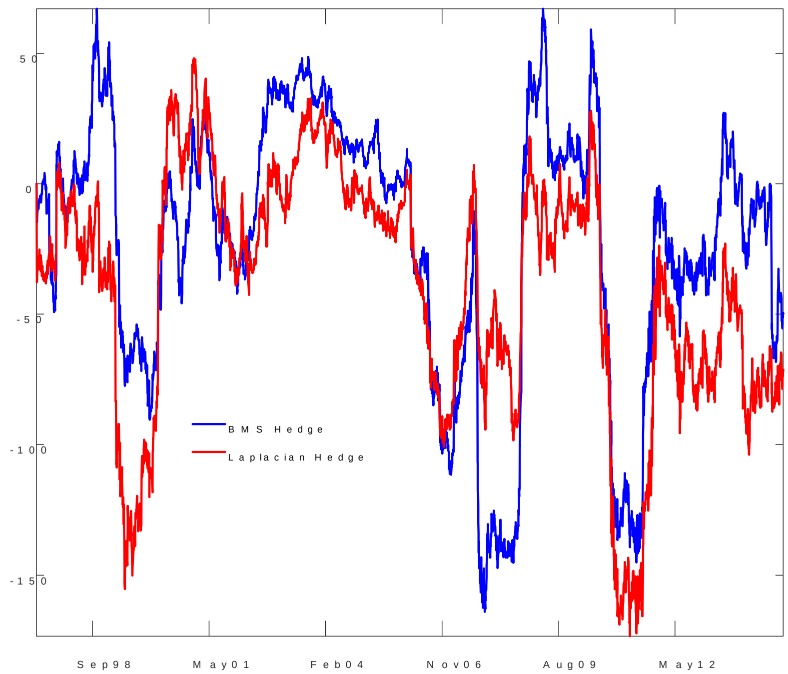

Figure 7: Valuations based on distorted expectations of the succeeding 252 days of daily PNL observations. The stress level used was 0.075.

values based on such alternative densities. The density we consider has the form

$$g(x) = \kappa \left[\mathbf{1}_{x>0} \exp\left(-c_p|x|^\alpha\right) + \mathbf{1}_{x<0} \exp\left(-c_n|x|^\alpha\right)\right]$$

where κ is a normalizing constant and c_p, c_n are calibrated to the variance and the possibly asymmetric probability that $x > 0$ differs from $1/2$. The stock price is given by the exponential of x adjusted to organize the correct risk neutral drift.

The difficulty with formulation here is that we do not have a closed form for the integration of the exponential against such a density. Numerical integrations could be performed but this takes matters too far from the basic advantages of working with implied volatilities. Our solution is to observe that we may approximate the exponential very well by

$$a(x) = \sum_{n=0}^{8} \frac{x^n}{n!}$$

in the interval $|x| \leq 2$ and we are comfortable with such a truncation of the density. So we work with $a(x)$ in the interval $[-2, 2]$.

To perform all required integrations we need to be able to integrate the form

$$H(b, c, n, \alpha) = \int_b^2 x^n e^{-cx^\alpha} dx, \;\; 0 \leq b \leq 2.$$

We define

$$\begin{aligned} H(b, c, n, \alpha) &= \int_{cb^\alpha}^{c2^\alpha} \left(\frac{u}{c}\right)^{\frac{n}{\alpha}} e^{-u} \frac{1}{\alpha c^{\frac{1}{\alpha}}} u^{1/\alpha - 1} du \\ &= \frac{\Gamma\left(\frac{n+1}{\alpha}\right)}{\alpha c^{\frac{n+1}{\alpha}}} \left[gammainc\left(cb^\alpha, \frac{n+1}{\alpha}, upper\right) - gammainc\left(c2^\alpha, \frac{n+1}{\alpha}, upper\right)\right]. \end{aligned}$$

The normalizing constant is then

$$\kappa = \begin{pmatrix} \frac{\Gamma\left(\frac{1}{\alpha}\right)}{\alpha c_p^{\frac{1}{\alpha}}} \left(1 - gammainc\left(c_p 2^\alpha, \frac{1}{\alpha}, upper\right)\right) \\ + \frac{\Gamma\left(\frac{1}{\alpha}\right)}{\alpha c_n^{\frac{1}{\alpha}}} \left(1 - gammainc\left(c_n 2^\alpha, \frac{1}{\alpha}, upper\right)\right) \end{pmatrix}^{-1}.$$

The mean is given by

$$\mu = \kappa(H(0, c_p, 1, \alpha) - H(0, c_n, 1, \alpha))$$

The noncentral moments are given by

$$E[x^n] = \kappa\left(H(0, c_p, n, \alpha) + (-1)^n H(0, c_n, 1, \alpha)\right)$$

The probability that $x > 0$ is

$$\eta = \frac{\frac{\Gamma\left(\frac{1}{\alpha}\right)}{\alpha c_p^{\frac{1}{\alpha}}} \left(1 - gammainc\left(c_p 2^\alpha, \frac{1}{\alpha}, upper\right)\right)}{\frac{\Gamma\left(\frac{1}{\alpha}\right)}{\alpha c_p^{\frac{1}{\alpha}}} \left(1 - gammainc\left(c_p 2^\alpha, \frac{1}{\alpha}, upper\right)\right) + \frac{\Gamma\left(\frac{1}{\alpha}\right)}{\alpha c_n^{\frac{1}{\alpha}}} \left(1 - gammainc\left(c_n 2^\alpha, \frac{1}{\alpha}, upper\right)\right)}$$

A later section presents the mapping, for a fixed value for α, from the σ, η parameterization to the parameters c_p, c_n and in the reverse direction.

8.0.1 LG Option Pricing

We now come to pricing a call option under this model given the parameters c_p, c_n, α valid at any maturity. For this we first need the convexity correction and define

$$S = S_0 \exp\left((r-q)\,t + \omega + x\right)$$

where x has the density $g(x)$.

We first compute ω as

$$\begin{aligned}
e^{-\omega} &= \left(\int_{-2}^{2}\sum_{n=0}^{8}\frac{x^n}{n!}g(x)dx\right) \\
&= \sum_{n=0}^{8}\left(\int_0^2 \frac{x^n}{n!}\kappa\exp\left(-c_p x^{\alpha}\right)dx + (-1)^n\int_0^2 \kappa\frac{x^n}{n!}\kappa\exp\left(-c_n x^{\alpha}\right)dx\right) \\
&= \kappa\left(\sum_{n=0}^{8} H(0,c_p,n,\alpha) + (-1)^n H(0,c_n,n,\alpha)\right).
\end{aligned}$$

We now consider a call option of strike K with payoff

$$\begin{aligned}
& (S_0\exp((r-q)t+\omega+x) - K)^+ \\
=\; & (S_0\exp((r-q)t+\omega)a(x) - K)^+
\end{aligned}$$

The relevant domain of integration is given by

$$\ln\left(\frac{S_0}{K}\right) + (r-q)t + \omega + x > 0$$

or

$$\begin{aligned}
x &> -b \\
b &= \min\left(\ln\left(\frac{S_0}{K}\right) + (r-q)t + \omega, 2\right).
\end{aligned}$$

The call price is given by

$$S_0 e^{-qt+\omega}\int_{-b}^{2} a(x)g(x)dx - e^{-rt}K\int_{-b}^{2} g(x)dx$$

For the second integral we get

$$\begin{aligned}
\int_{-b}^{2} g(x)dx &= \mathbf{1}_{b<0}\kappa H(-b,c_p,0,\alpha) + \mathbf{1}_{b>0}\kappa\left(H(0,c_p,0,\alpha) + H(0,c_n,0,\alpha) - H(b,c_n,0,\alpha)\right) \\
&= \mathbf{1}_{b<0}\kappa H(-b,c_p,0,\alpha) + \mathbf{1}_{b>0}\left(1 - \kappa H(b,c_n,0,\alpha)\right)
\end{aligned}$$

For the first integral we have

$$\int_{-b}^{2} a(x)g(x)dx = \mathbf{1}_{b<0}\sum_{n=0}^{8}\frac{\kappa H(-b,c_p,n,\alpha)}{n!} + \mathbf{1}_{b>0}\left(\sum_{n=0}^{8}\frac{\kappa H(0,c_p,n,\alpha)}{n!} + \sum_{n=0}^{8}\frac{\kappa(-1)^n\left(H(0,c_n,n,\alpha)-H(b,c_n,n,\alpha)\right)}{n!}\right) \quad (11)$$

For the put option we use put call parity.

8.1 Laplace Gauss Delta

The Call price is given by

$$C = S_0 e^{-qt+\omega}\int_{-b}^{2} e^x g(x)dx - Ke^{-rt}\int_{-b}^{2} g(x)dx$$

For the call delta we get

$$\frac{\partial C}{\partial S_0} = e^{-qt+\omega}\int_{-b}^{2} e^x g(x)dx + \frac{\partial b}{\partial S_0}e^{-b}g(-b) - Ke^{-rt}\frac{\partial b}{\partial S_0}g(-b)$$

Now we have that

$$S_0 e^{(r-q)t+\omega-b} = K$$

and hence the last two terms cancel. For the first we write

$$\frac{\partial C}{\partial S_0} = e^{-qt}\frac{\int_{-b}^{2} e^x g(x)dx}{\int_{-2}^{2} e^x g(x)dx}$$

and hence the delta is the probability that the call is in the money under the so called share measure. For the put delta we use put call parity and write

$$\begin{aligned}\frac{\partial P}{\partial S_0} &= \frac{\partial C}{\partial S_0} - e^{-qt} \\ &= -e^{-qt}\left(1 - \frac{\int_{-b}^{2} e^x g(x)dx}{\int_{-2}^{2} e^x g(x)dx}\right)\end{aligned}$$

Hence both deltas are determined on evaluating

$$A = \int_{-b}^{2} e^x g(x)dx$$

for which we uspe 11.
We then have that

$$\begin{aligned}\frac{\partial C}{\partial S_0} &= e^{-qt}e^{\omega}A \\ \frac{\partial P}{\partial S_0} &= -e^{-qt}\left(1 - e^{\omega}A\right).\end{aligned}$$

8.2 Closed form for the mappings from σ, η to c_p, c_n given α and back

We may treat the integral from 2 to infinity as zero. The conditions for calibration are then

$$\begin{aligned}\eta &= \frac{\frac{\Gamma\left(\frac{1}{\alpha}\right)}{\alpha c_p^{\frac{1}{\alpha}}}}{\frac{\Gamma\left(\frac{1}{\alpha}\right)}{\alpha c_p^{\frac{1}{\alpha}}} + \frac{\Gamma\left(\frac{1}{\alpha}\right)}{\alpha c_n^{\frac{1}{\alpha}}}} \\ &= \frac{1}{1 + \left(\frac{c_p}{c_n}\right)^{\frac{1}{\alpha}}}\end{aligned}$$

which yields

$$\frac{c_p}{c_n} = \left(\frac{1}{\eta} - 1\right)^{\alpha}.$$

The variance condition is

$$\sigma^2 t = \frac{\Gamma\left(\frac{3}{\alpha}\right)}{\alpha c_p^{\frac{3}{\alpha}}} + \frac{\Gamma\left(\frac{3}{\alpha}\right)}{\alpha c_n^{\frac{3}{\alpha}}} - \left(\frac{\Gamma\left(\frac{2}{\alpha}\right)}{\alpha c_p^{\frac{2}{\alpha}}} - \frac{\Gamma\left(\frac{2}{\alpha}\right)}{\alpha c_n^{\frac{2}{\alpha}}}\right)^2$$

which is the expectation of the square less the square of the mean. This may be written as

$$\frac{\Gamma\left(\frac{3}{\alpha}\right)}{\alpha}\left(\frac{1}{c_p^{\frac{3}{\alpha}}} + \frac{1}{c_n^{\frac{3}{\alpha}}}\right) - \frac{\Gamma\left(\frac{2}{\alpha}\right)^2}{\alpha^2}\left(\frac{1}{c_p^{\frac{4}{\alpha}}} + \frac{1}{c_n^{\frac{4}{\alpha}}} - \frac{2}{(c_p c_n)^{\frac{2}{\alpha}}}\right)$$

We know for

$$\beta = \left(\frac{1}{\eta} - 1\right)^{\alpha}$$

that

$$c_p = \beta c_n$$

and letting $x = c_n$ we get

$$\sigma^2 t = \frac{\Gamma\left(\frac{3}{\alpha}\right)}{\alpha}\left(\frac{1}{\beta^{\frac{3}{\alpha}}} + 1\right)\frac{1}{x^{\frac{3}{\alpha}}} - \frac{\Gamma\left(\frac{2}{\alpha}\right)^2}{\alpha^2}\left(\frac{1}{\beta^{\frac{4}{\alpha}}} + 1 - \frac{2}{\beta^{\frac{2}{\alpha}}}\right)\frac{1}{x^{\frac{4}{\alpha}}}$$

For η near $1/2$ and β near unity we may ignore the square of the mean term and write

$$\begin{aligned}c_n &= \left(\frac{\Gamma\left(\frac{3}{\alpha}\right)}{\alpha}\left(\frac{1}{\beta^{\frac{3}{\alpha}}} + 1\right)\frac{1}{\sigma^2 t}\right)^{\frac{\alpha}{3}} \\ c_p &= \beta c_n \\ \beta &= \left(\frac{1}{\eta} - 1\right)^{\alpha}\end{aligned}$$

In the opposite direction we write

$$
\begin{aligned}
\eta &= \frac{1}{1+\left(\frac{c_p}{c_n}\right)^{\frac{1}{\alpha}}} \\
\sigma^2 t &= \frac{\Gamma\left(\frac{3}{\alpha}\right)}{\alpha c_p^{\frac{3}{\alpha}}} + \frac{\Gamma\left(\frac{3}{\alpha}\right)}{\alpha c_n^{\frac{3}{\alpha}}}
\end{aligned}
$$

9 Experiments with LG and Adapted hedging

We report here on the result of constructing 10 hedges using Black, Merton Scholes, the symmetric Laplace Gauss with powers .25, .5, and .75 and then the same three biased down to an up probability of .45 and an up probability of .55. In addition we evaluate the bid price for the change in net worth over the last quarter at a stress level of 0.1 and put on the hedge the next day using the strategy with the largest bid price. The product hedged was the stylized MSUSSPVP from March 15 2006 to March 16 2016. Figure 8 presents the time path of the index value for the Black Merton Scholes, Laplace Gauss power 0.75 in the symmetric and two biased cases for down and up along with the adpated hedge. It clear in this case that the adapted hedge dominates all the other fixed hedging strategies.

10 Conclusion

The paper replaces the exponential of squared returns embedded in Gaussian densities and their associated thin tails with the absolute value of returns to form Laplacian and exponentially tilted Laplacian densities at unit time. Densities at other time points are obtained by scaling and stochastic processes consistent with these marginals are identified. Aside from local volatility we identify the densities as consistent with the difference of compound exponential processes taken at log time and scaled by the square root of time. The underlying process has a single parameter, the constant variance rate of the process. The unit time density is also that of a normal distribution with a exponentially distributed variance.

Delta hedging strategies based on Laplacian and Asymmetric Laplacian implied volatilities are then formulated and compared with Black Merton Scholes implied volatility hedging. The hedging strategies are implemented by in stylized businesses represented by dynamic volatility indexes. The Laplacian hedge is seen to be smoother for the skew trade of the index MSUSMSDS. It also performs better through the financial crisis for the index MSUSSPVP. Finally an adapted hedge is formulated by evaluating a hedge performance measure on past quarterly data with the best hedge used for the next day. This adapted hedge dominates all fixed hedge strategies for the product MSUSSPVP.

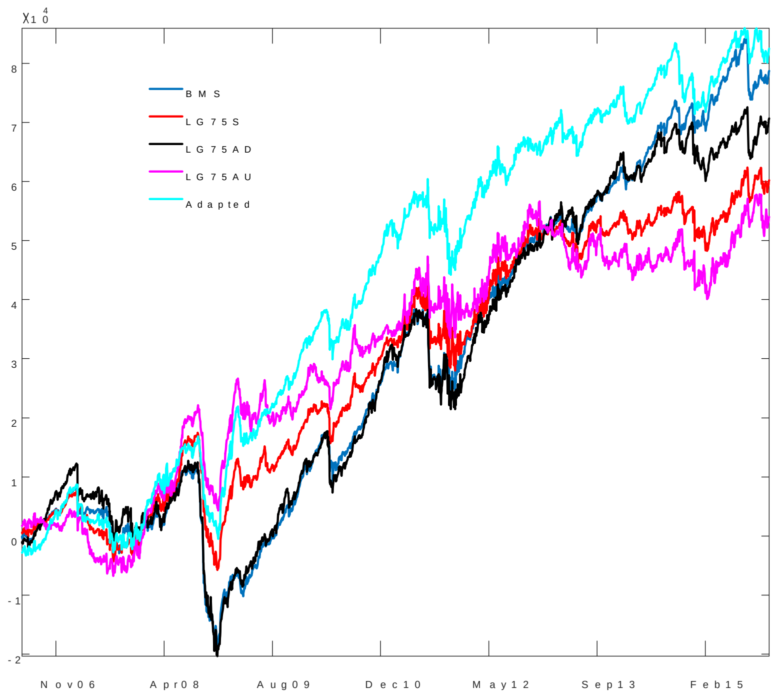

Figure 8: Black Merton Scholes, Laplace Gauss Symmetric, Asymmetric Down and Asymmetric Up along with the adapted hedge.

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